THE RATIO TEST

(B.Sc.-II, Paper-III)

Group- B

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The Ratio Tests

Theorem (D'Alembert's Ratio Test) :→

Let Σa_n be a series of positive terms such that,

(i) $\dfrac{a_{n+1}}{a_n} < K < 1$, where K is a constant and $n \geq m$ then Σa_n is convergent.

(ii) $\dfrac{a_{n+1}}{a_n} \geq 1$, for all $n \geq m$ then Σa_n is divergent.

Proof:→ Without any loss of generality, we can assume the condition to be true for $n \geq 1$.
($\because$ convergence or divergence is not affected by omission of finite number of terms)

(i) Let $\dfrac{a_{n+1}}{a_n} < K < 1$, for $n \geq 1$.

$$\therefore \frac{a_n}{a_1} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_{n-2}}{a_{n-3}} \cdots\cdots\cdots \frac{a_3}{a_2} \cdot \frac{a_2}{a_1}$$

$$< K \cdot K \cdot K \cdots\cdots\cdots, (n-1) \text{ times}$$

$$\therefore \frac{a_n}{a_1} < K^{n-1} \quad \text{for } n \geq 1.$$

$$\Rightarrow \quad a_n < a_1 K^{n-1}, \quad \text{for all } n \geq 1.$$

But $\sum a_1 k^{n-1}$ is a G.P. series with common ratio $k < 1$.

$\therefore$ It is convergent

$\therefore$ By comparision test

$\sum a_n$ is convergent

(ii) Let $\frac{a_{n+1}}{a_n} \geq 1$, for $n \geq 1$.

In this case

$$\frac{a_2}{a_1} \geq 1, \quad \frac{a_3}{a_2} \geq 1, \quad \frac{a_4}{a_3} \geq 1, \ \ldots\ldots$$

$$\therefore a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$$

$$\therefore a_1 + a_2 + \cdots + a_n \geq n a_1$$

$$\therefore \quad S_n \geq n a_1, \quad \text{where } S_n = a_1 + a_2 + \cdots + a_n$$

$$\therefore \lim_{n\to\infty} S_n = \infty \quad \left(\because \lim_{n\to\infty} n a_1 = \infty\right)$$

$\therefore$ The series is divergent.

Theorem (D'Alembert Ratio test, limit form)

statement: $\rightarrow$ Let $\sum a_n$ be a series of positive terms such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$,

then (i) $\sum a_n$ is convergent if $l < 1$.

(ii) $\sum a_n$ is divergent if $l > 1$.

Remark: $\rightarrow$ The test fails if $l = 1$.

Proof: → (i) Let $l < 1$.

We can choose ϵ such that $l + \epsilon < 1$.

Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$.

∴ There exists a natural number N, s.t.

$$l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon, \text{ for all } n \geq N.$$

$$\therefore \frac{a_{n+1}}{a_n} < l + \epsilon < 1, \text{ for all } n \geq N.$$

∴ By above theorem, Σa_n is convergent.

proved.

(ii) Let $l > 1$.

We can choose ϵ, such that $l - \epsilon > 1$

Since $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$.

∴ There exists a natural number N s.t.

$$l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon, \text{ for all } n \geq N$$

$$\therefore \frac{a_{n+1}}{a_n} > l - \epsilon > 1, \text{ for all } n \geq N.$$

∴ By above theorem, Σa_n is divergent.

proved.

Remark: → (i) Let $\sum a_n = \sum \frac{1}{n}$

Then $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$

$\therefore$ The series $\sum a_n$ is divergent.

(ii) Let $\sum b_n = \sum \frac{1}{n^2}$

The series is ~~divergent~~ convergent ($\because p = 2 > 1$)

But $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1.$

Hence the series may converge or diverge when $l = 1$.

Example ①: → Show that the series

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots$$

is convergent for all positive values of x.

Solution: → Here $a_n = \frac{x^{n-1}}{\lfloor n-1}$,

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^n}{\lfloor n} \times \frac{\lfloor n-1}{x^{n-1}}$$

$$= \lim_{n\to\infty} \frac{x}{n} = 0 < 1.$$

$\therefore$ By D'Alembert's ratio test,,

The given series is convergent for all positive values of x.

proved.

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Example 2:→ Test the following series is convergent or divergent:

$$2x + \frac{2x^2}{8} + \frac{4x^3}{27} + \cdots\cdots + \frac{(n+1)}{n^3}x^n + \cdots \quad (x>0).$$

Solution:→

Here $a_n = \frac{(n+1)x^n}{n^3}$. and $a_{n+1} = \frac{(n+2)}{(n+1)^3}\cdot x^{n+1}$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)x^{n+1}}{(n+1)^3} \times \frac{n^3}{(n+1)\cdot x^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 \cdot \left(\frac{n+2}{n+1}\right) \cdot x$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^3} \cdot \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)} \cdot x$$

$$= \lim_{n\to\infty} x = x$$

Hence by D'Alembert's test the given series is convergent if $0<x<1$. and ~~the~~ the series is divergent if $x>1$.

At $x=1$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. and the test fails.

At $x=1$, $a_n = \frac{n+1}{n^3}$

Let $b_n = \frac{1}{n^2}$

$$\because \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+1)}{n^3} = 1 \ (\neq 0)$$

$\because \sum b_n = \sum \frac{1}{n^2}$ is convergent

$\therefore \sum a_n$ is convergent

Therefor the given series is convergent when $0 < x \leq 1$ and divergent if $x > 1$.

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Example ③ :→ Test the convergence of the series

$$\sum \frac{n^2-1}{n^2+1} x^n \quad (x>0)$$

Solution :→ Let the nth term of the given series be denoted by a_n.

$$\therefore a_n = \frac{n^2-1}{n^2+1} \cdot x^n, \quad a_{n+1} = \frac{(n+1)^2 - 1}{(n+1)^2 + 1} \cdot x^{n+1}$$

$$\therefore \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\{(n+1)^2 - 1\} \cdot x^{n+1}}{\{(n+1)^2 + 1\}} \times \frac{\{n^2+1\}}{\{n^2-1\} \cdot x^n}$$

$$= \lim_{n \to \infty} \frac{n^2+2n}{n^2+2n+2} \times \frac{n^2+1}{n^2-1} \cdot x$$

$$= \lim_{n \to \infty} \left(\frac{1 + \frac{2}{n}}{1 + \frac{2}{n} + \frac{2}{n^2}} \right) \left(\frac{1 + \frac{1}{n^2}}{1 - \frac{1}{n^2}} \right) \cdot x$$

$$\therefore \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = x$$

Hence by D'Alembert's ratio test, the given series is convergent if $0 < x < 1$ and the series is divergent if $x > 1$.

At $x = 1$, the given series becomes $\sum \frac{n^2-1}{n^2+1} = \sum a_n$

$\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1}$

$$= \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1 \neq 0.$$

$\therefore$ At $x = 1$, the given series is divergent.

$\therefore$ The given series is convergent if $0 < x < 1$ and divergent if $x \geq 1$.

Example ④ Test the convergence of the series $\sum a_n = \sum \frac{x^n}{1+n^2}$, where $x > 0$.

Solution:→ Here $a_n = \frac{x^n}{1+n^2}$ & $a_{n+1} = \frac{x^{n+1}}{1+(n+1)^2}$

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+n^2}{1+(n+1)^2} \cdot x$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2}+1}{\frac{1}{n^2}+\left(1+\frac{1}{n}\right)^2} \cdot x = x$$

$\therefore$ By D'Alembert's ratio test, the series is convergent if $0 < x < 1$ and diverges if $x > 1$.

If $x = 1$, test fails.

Let $b_n = \frac{1}{n^2}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1 \cdot (\neq 0)$

$\therefore$ By comparision test

$\sum a_n$ is convergent. ($\because \sum b_n = \sum \frac{1}{n^2}$ is convergent)

Thus the given series is convergent if $0 < x \leq 1$ and divergent if $x > 1$.

Thank you